

# Traversable Wormholes Construction in (2+1) Gravity

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Wormholes have been always an interesting object in gravity theories. In this paper we make a little review of the principal properties of these objects and the exotic matter they need to exist. Then, we obtain two specific solutions in the formalism of (2+1)-dimensional gravity with negative cosmological constant. The obtained geometries correspond to traversable wormholes with an exterior geometry correspondent to the well known BTZ black hole solution. We also discuss the distribution of exotic matter that these wormholes need.

## I. INTRODUCTION

20 years ago Morris-Thorne[3] propose the traversable wormholes as a teaching tool for General Relativity. In 1995, the well known book of Visser collects all the work done in this area on the last century. Today, wormholes are one of the most studied solutions of Einstein's field equations because of their characteristics and their relation with the existence of a special type of matter that would violate the energy conditions, now called *exotic matter*.

On the other side, the  $(2 + 1)$  dimensional gravity is a covariant theory of spacetime geometry that has a great simplicity when compared with General Relativity, and this made of it a good theory to study some quantum aspects of gravity. The most interesting solution of this theory, is the black hole metric found by Banados, Teitelboim and Zanelli (BTZ Black Hole)[16, 17] in a universe with a negative cosmological constant.

Some years ago, some authors are interested on wormholes in  $(2 + 1)$  dimensional gravity. Delgaty et. al. [21] made an analysis of the characteristic of one specific wormhole in universes with cosmological constant. Aminneborg et. al. [19] compares the properties of wormholes with the characteristic of black holes while Kim et. al. [20] give two specific solutions taking  $(2 + 1)$  dimensional gravity with a dilatonic field.

In this paper we give two specific wormhole solutions in  $(2 + 1)$  gravity obtained joining two spacetime manifolds following the work of Lemos et. al. [5]. The exterior of this wormholes correspond to the BTZ metric without electric charge and without angular momentum. We also show the matter distribution that is needed to maintain the solutions.

## II. STRUCTURE EQUATIONS FOR THE WORMHOLE

### A. Traversable Wormholes Properties

In order to restrain the possible solutions to traversable wormholes we must impose some conditions [3]:

1. The metric must be a solution of the field equations at every spacetime point.
2. Metric must have spherical symmetry and must be static (i.e. time independent).
3. Solution must have a "throat" that connects two spacetime regions. In the  $(2 + 1)$  case, the exterior spacetime must correspond to BTZ solution.
4. Metric must not have event horizons (it would prevent two way travel)
5. Tidal forces must be small or null (we want to be possible to travel into the wormhole)
6. The time needed to cross the wormhole must be reasonable.

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These conditions will be imposed in order to obtain the specific solutions.

The wormhole metric must be spherically symmetric and time independent. Thus, in usual spherical coordinates  $(t, r, \varphi)$ , we have the general metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{1}{1 - \frac{b(r)}{r}} dr^2 + r^2 d\varphi^2, \quad (1)$$

where  $\Phi(r)$ ,  $b(r)$  are arbitrary functions of the radial coordinate  $r$ , that will be restrained by the imposed conditions. The function  $\Phi(r)$  is known as “*redshift function*”, while  $b(r)$  is the “*shape function*”.

The field equations with a cosmological constant can be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad (2)$$

where  $g_{\mu\nu}$  is the metric,  $\Lambda$  is the cosmological constant,  $T_{\mu\nu}$  is the stress-energy tensor and  $G_{\mu\nu}$  is the Einstein tensor defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (3)$$

with  $R_{\mu\nu}$  the Ricci tensor. Remember that greek indice take the values 0, 1, 2.

Using the metric (1), the Einstein tensor has components

$$\begin{aligned} G_{tt} &= \frac{1}{2r^3} e^{2\Phi} [-b + rb'] \\ G_{rr} &= \frac{\Phi'}{r} \\ G_{\varphi\varphi} &= \frac{1}{2} \left[ \Phi' (b - rb') + 2r (r - b) \left( (\Phi')^2 + \Phi'' \right) \right], \end{aligned} \quad (4)$$

where primes represent derivative with respect to the radial coordinate  $r$ .

## B. Change of Basis

The Einstein tensor obtained above is defined using the  $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\varphi)$  triad associated with the coordinates  $t, r, \varphi$ . However, we can choose any coordinate system and, in this case, is useful to consider a set of orthonormal vectors as basis. These correspond to the reference system of an observer that remains at rest in the  $(t, r, \varphi)$  system; i.e. with  $r, \varphi$  constant.

We will denote this triad by  $(\mathbf{e}_{\hat{t}}, \mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\varphi}})$ ; and its relation with the original  $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\varphi)$  is

$$\mathbf{e}_{\hat{t}} = e^{-\Phi} \mathbf{e}_t \quad (5)$$

$$\mathbf{e}_{\hat{r}} = \left( 1 - \frac{b}{r} \right)^{1/2} \mathbf{e}_r \quad (6)$$

$$\mathbf{e}_{\hat{\varphi}} = \frac{1}{r} \mathbf{e}_\varphi \quad (7)$$

It is important to note that in the new system, the metric tensor is

$$g_{\alpha\beta} = \mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

and all the tensor will change their components. For example, the field equation will take the form

$$G_{\hat{\mu}\hat{\nu}} + \Lambda \eta_{\hat{\mu}\hat{\nu}} = T_{\hat{\mu}\hat{\nu}}. \quad (9)$$

where the new Einstein tensor is now given by

$$G_{\hat{t}\hat{t}} = \frac{1}{2r^3} [b'r - b] \quad (10)$$

$$G_{\hat{r}\hat{r}} = \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} \quad (11)$$

$$G_{\hat{\varphi}\hat{\varphi}} = \frac{1}{2r^2} \left[ \Phi' (b - rb') + 2r(r - b) \left( (\Phi')^2 + \Phi'' \right) \right] \quad (12)$$

### III. STRESS- ENERGY TENSOR

In order to obtain a traversable wormhole, we need a non null Stress-Energy tensor. Since the field equations (9) tell us that the stress-energy tensor is proportional to Einstein tensor, they must have the same algebraic structure, i. e. that the non zero components of  $T_{\hat{\mu}\hat{\nu}}$  must be  $T_{\hat{t}\hat{t}}$ ,  $T_{\hat{r}\hat{r}}$  and  $T_{\hat{\varphi}\hat{\varphi}}$ .

In the orthonormal basis  $(\mathbf{e}_{\hat{t}}, \mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\varphi}})$  related with the inertial reference system of an static observer, the components of the stress-energy tensor take an immediate interpretation,

$$\begin{aligned} T_{\hat{t}\hat{t}} &= \rho(r) \\ T_{\hat{r}\hat{r}} &= -\tau(r) \\ T_{\hat{\varphi}\hat{\varphi}} &= p(r), \end{aligned} \quad (13)$$

where  $\rho(r)$  is the mass-energy density,  $\tau(r)$  is the radial tension per unit area (i.e. the negative of the radial pressure,  $\tau(r) = -p_r(r)$ ) and  $p(r)$  is the tangential pressure.

Sometimes its interesting to write the cosmological constant term in the field equations (9) as

$$T_{\hat{\mu}\hat{\nu}}^{(vac)} = -\Lambda \eta_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & -\Lambda & 0 \\ 0 & 0 & -\Lambda \end{bmatrix}, \quad (14)$$

and then the field equations will be

$$G_{\hat{\mu}\hat{\nu}} = \left( T_{\hat{\mu}\hat{\nu}} + T_{\hat{\mu}\hat{\nu}}^{(vac)} \right) \quad (15)$$

$$G_{\hat{\mu}\hat{\nu}} = \bar{T}_{\hat{\mu}\hat{\nu}}, \quad (16)$$

where  $\bar{T}_{\hat{\mu}\hat{\nu}} = T_{\hat{\mu}\hat{\nu}} + T_{\hat{\mu}\hat{\nu}}^{(vac)}$  is the total stress-energy tensor. Therefore, we can define the functions  $\bar{\rho}(r)$ ,  $\bar{\tau}(r)$  and  $\bar{p}(r)$  by

$$\begin{aligned} \bar{\rho}(r) &= \rho(r) + \Lambda \\ \bar{\tau}(r) &= \tau(r) + \Lambda \\ \bar{p}(r) &= p(r) - \Lambda \end{aligned} \quad (17)$$

### IV. SOLVING THE FIELD EQUATIONS

Using the Einstein tensor given by (10-12) and the stres-energy tensor (13), we can obtain the field equations,

$$\rho(r) = \frac{1}{2r^3} [b'r - b] - \Lambda \quad (18)$$

$$\tau(r) = -\left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} - \Lambda \quad (19)$$

$$p(r) = \frac{1}{2r^2} \left[ \Phi'(b - rb') + 2r(r - b) \left( (\Phi')^2 + \Phi'' \right) \right] + \Lambda. \quad (20)$$

Taking the derivative of equation (19) with respect to  $r$  we have

$$\tau'(r) = - \left( 1 - \frac{b}{r} \right) \frac{\Phi''}{r} + \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r^2} + \frac{b'r - b}{r^3} \Phi'. \quad (21)$$

Using equations (18-20) to eliminate  $b'$  and  $\Phi''$  we obtain

$$\tau'(r) = (\rho - \tau) \Phi' - \frac{p + \tau}{r} \quad (22)$$

Equations (18), (20) and (22) are three differential equations that correlate the five unknown functions  $b, \Phi, \rho, \tau$  and  $p$ .

Now, the usual way to solve these equations is to assume a specific kind of matter and energy. The corresponding state equation gives a relationship between the tension and the density function,  $\tau(\rho)$ , and between the pressure and density,  $p(\rho)$ . Therefore, we got five equations for five unknown functions, and we can find the form of the spacetime manifold, i.e. we can obtain the functions  $b(r)$  and  $\Phi(r)$ .

For wormholes, we proceed in a different way: we impose the conditions on the geometry of the spacetime manifold (i. e. we impose the functions  $b(r)$  and  $\Phi(r)$ ), and using the field equations we obtain the needed matter-energy distribution for that geometry.

### A. Geometry of the Wormhole

The wormhole metric given by (1), considered for a fixed time  $t$  is

$$ds^2 = \frac{dr^2}{1 - \frac{b}{r}} + r^2 d\varphi^2. \quad (23)$$

If we make an embedding of this metric in the three-dimensional euclidean space with cylindrical coordinates,

$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2, \quad (24)$$

we obtain the equation for the embedding surface,

$$\frac{dz}{dr} = \pm \left( \frac{r}{b} - 1 \right)^{-1/2}. \quad (25)$$

To obtain a wormhole geometry, the solution must have a minimum radius called “throat”,  $r = b(r) = r_m$ . At the throat the embedded surface is vertical, i.e.  $\frac{dz}{dr} \rightarrow \infty$ . On the other hand, far from the mouth of the wormhole, the space is asymptotically flat, i.e.  $\frac{dz}{dr} \rightarrow 0$ .

Since the wormhole metric must be connected smoothly with the exterior spacetime, the throat must flare out. this condition can be written in terms of the embedding function as

$$\frac{d^2 r}{dz^2} > 0. \quad (26)$$

Using equation (25) we have

$$\frac{dr}{dz} = \pm \left( \frac{r}{b} - 1 \right)^{1/2}, \quad (27)$$

and differentiating with respect to  $z$  we obtain

$$\frac{d^2 r}{dz^2} = \pm \frac{1}{2} \left( \frac{b - rb'}{b^2} \right). \quad (28)$$

Therefore, the flare out condition is

$$\frac{d^2 r}{dz^2} = \frac{b - rb'}{2b^2} > 0 \quad \text{at the throath or near} \quad (29)$$

Now, in order to assure that the wormhole permits inside and outside travel, we need that there will be no event horizon. For static metrics, horizons correspond to non singular surfaces at which

$$g_{tt} = -e^{2\Phi} \rightarrow 0.$$

Then, in order to assure traversability, we need that the function  $\Phi(r)$  be finite at every point.

## V. PROPERTIES OF THE STRESS-ENERGY TENSOR

If we define the adimensional function

$$\varsigma = \frac{\tau - \rho}{|\rho|}, \quad (30)$$

the wormhole field equations (18) and (19) give

$$\varsigma = \frac{\tau - \rho}{|\rho|} = \frac{-2r^2 \left(1 - \frac{b}{r}\right) \Phi' - (b'r - b)}{|b'r - b - 2r^3 \Lambda|}. \quad (31)$$

To obtain a wormhole, we must demand that the inside metric joins soomthly with the outside metric ( that corresponds to BTZ metrc), and then we impose the flare out condition described above. This condition is given by

$$\frac{d^2 r}{dz^2} = \frac{b - rb'}{2b^2} > 0 \quad \text{at the throath or near it.} \quad (32)$$

Then, equation (31) is

$$\varsigma = \frac{\tau - \rho}{|\rho|} = \frac{2b^2}{|2b^2 \frac{d^2 r}{dz^2} + 2r^3 \Lambda|} \frac{d^2 r}{dz^2} - 2 \left(1 - \frac{b}{r}\right) \frac{r^2 \Phi'}{|2b^2 \frac{d^2 r}{dz^2} + 2r^3 \Lambda|}. \quad (33)$$

Near the throat we have  $\left(1 - \frac{b}{r}\right) \Phi' \rightarrow 0$ . Therefore the flare out condition implicates

$$\varsigma_m = \frac{\tau_m - \rho_m}{|\rho_m|} > 0, \quad (34)$$

where de index  $m$  indicates that we are evaluating at the thorath or near it.

The condition  $\tau_m > \rho_m$  is imposed by (34) and any material that satisfies the property ( $\tau_m > \rho_m > 0$ ) is called “exotic” and will violate the *energy conditions*[1].

## VI. CONSTRUCTION OF WORMHOLES

In order to construct the wormholes we use the equations that relate  $b, \Phi, \rho, \tau$  y  $p$ ,

$$\begin{aligned} \rho(r) &= \frac{1}{2r^3} [b'r - b] - \Lambda \\ \tau(r) &= - \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} - \Lambda \\ p(r) &= \frac{1}{2r^2} \left[ \Phi' (b - rb') + 2r(r - b) \left( (\Phi')^2 + \Phi'' \right) \right] + \Lambda \\ \tau'(r) &= (\rho - \tau) \Phi' - \frac{p + \tau}{r}. \end{aligned} \quad (35)$$

Since we work with a cosmological constant we will distinguish between the inside solution( i.e.  $r < a$ , con  $\Lambda_{int}$ ) and the outside solution (i.e.  $r > a$ , con  $\Lambda_{ext}$ ).

### A. Interior Solution

The interior solution must have the wormhole form

$$ds^2 = -e^{2\Phi^{int}(r)} c^2 dt^2 + \frac{1}{1 - \frac{b^{int}(r)}{r}} dr^2 + r^2 d\varphi^2. \quad (36)$$

To find explicitly the functions  $\Phi_{int}(r)$  and  $b_{int}(r)$  inside ( $r < a$ ), we will use  $\Lambda_{int}$  in the equations (35). (We will not use the index  $^{int}$  in the functions  $\Phi$  y  $b$ )

$$\rho(r) = \frac{1}{2r^3} [b'r - b] - \Lambda_{int} \quad (37)$$

$$\tau(r) = - \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} - \Lambda_{int} \quad (38)$$

$$p(r) = \frac{1}{2r^2} \left[ \Phi'(b - rb') + 2r(r - b) \left( (\Phi')^2 + \Phi'' \right) \right] + \Lambda_{int} \quad (39)$$

In the equation for the tension, we see that at the throat ( $b(r_m) = r_m$ ), we have

$$\tau(r_m) = - \left(1 - \frac{r_m}{r_m}\right) \frac{\Phi'}{r_m} - \Lambda_{int} \quad (40)$$

$$\tau(r_m) = -\Lambda_{int}. \quad (41)$$

i. e. that the radial tension at the throat is positive for holes with  $\Lambda_{int} < 0$  and is negative (i. e. a pressure) for holes with  $\Lambda_{int} > 0$ . On the other side, it is interesting to note that the total radial tension at the throat is zero ,

$$\overline{\tau}(r_m) = \tau(r_m) + \Lambda_{int} = 0. \quad (42)$$

### B. Exterior Solution

In the exterior of the wormhole ( $r > a$ ) we consider a vacuum spacetime geometry, i. e. a null stress-energy tensor  $T_{\hat{\mu}\hat{\nu}} = 0$ . This means

$$\rho(r) = \tau(r) = p(r) = 0. \quad (43)$$

However, we may have a non null exterior cosmological constant  $\Lambda_{ext}$ . Equations (35) will be now

$$\begin{aligned} 0 &= \frac{1}{2r^3} [b'r - b] - \Lambda_{ext} \\ 0 &= - \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} - \Lambda_{ext} \\ 0 &= \frac{1}{2r^2} \left[ \Phi'(b - rb') + 2r(r - b) \left( (\Phi')^2 + \Phi'' \right) \right] + \Lambda_{ext}. \end{aligned} \quad (44)$$

Solving this equations we obtain the exterior solution[1],

$$ds^2 = - \left( -M - \Lambda_{ext} r^2 \right) dt^2 + \frac{dr^2}{(-M - \Lambda_{ext} r^2)} + r^2 d\varphi^2. \quad (45)$$

If the cosmological constant is negative  $\Lambda_{ext} < 0$ , we can write it, following Banados et. al.[16, 18], as

$$\Lambda_{ext} = -\frac{1}{l^2}, \quad (46)$$

and the exterior solution gives the usual BTZ black hole metric,

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2}\right)} + r^2 d\varphi^2. \quad (47)$$

Note that this solution have singularities at the radii

$$r_{\pm} = \pm\sqrt{Ml}. \quad (48)$$

The outside singularity  $r_+$  corresponds to the event horizon for the black hole and in order to satisfy the traversability conditions, we must impose  $a > r_+$ .

### C. Junction Conditions

In order to join the interior and exterior metrics we consider the boundary surface  $S$  that connects them. The first condition is that the metric must be continuous at  $S$ , i. e.  $g_{\mu\nu}^{int}|_S = g_{\mu\nu}^{ext}|_S$ .

However, this condition is not enough to make the junction. The Darmois-Israel formalism impose the continuity of the second fundamental form (extrinsic curvature) at the surface  $S$ . But, when the spacetime is spherically symmetric, the second condition can be done directly with the field equations.

With these conditions, we will find the stress-energy density at surface  $S$  needed to made the junction between the exterior and interior regions. When there is no stress-energy terms at  $S$ , we say that this is a *boundary surface*, while when we have som stress-energy terms we call it a *thin-shell*.

#### 1. Continuity of the metric

Since both the inside and outside metrics are spherically symmetric, then the continuity of the metric condition  $g_{\mu\nu}^{int}|_S = g_{\mu\nu}^{ext}|_S$  is immediate for the  $g_{\varphi\varphi}$  component. For the  $t$  and  $r$  components we impose

$$\begin{aligned} g_{tt}^{int}|_{r=a} &= g_{tt}^{ext}|_{r=a} \\ g_{rr}^{int}|_{r=a} &= g_{rr}^{ext}|_{r=a}. \end{aligned} \quad (49)$$

Using equations (36) and (45), the continuity conditions are

$$e^{2\Phi(a)} = -M + \frac{a^2}{l^2} \quad (50)$$

$$1 - \frac{b(a)}{a} = -M + \frac{a^2}{l^2}, \quad (51)$$

that can be written as

$$\Phi(a) = \frac{1}{2} \ln \left( -M + \frac{a^2}{l^2} \right) \quad (52)$$

$$b(a) = (1 + M) a - \frac{a^3}{l^2}. \quad (53)$$

Last equation let us obtain an expresion for the wormhole mass,

$$M = \frac{b(a)}{a} + \frac{a^2}{l^2} - 1. \quad (54)$$

## 2. Field Equations

To complete the junction of exterior and interior metrics we will use the field equations (2). We also suppose that static observers inside will feel null tidal forces, i.e.  $\Phi^{int} = \text{constant}$ , and therefore we have  $\Phi'^{int} = 0$ .

If we have a thin-shell, the components of the stress-energy tensor is non zero at the surface  $S$  and we can write them as proportional to the Dirac's delta function,

$$T_{\hat{\mu}\hat{\nu}} = t_{\hat{\mu}\hat{\nu}} \delta(\hat{r} - \hat{a}), \quad (55)$$

where  $\hat{r} = \sqrt{g_{rr}}r$  is the proper radial distance measured inside the thin-shell. To obtain the components  $t_{\hat{\mu}\hat{\nu}}$  we must use

$$\int_{int}^{ext} G_{\hat{\mu}\hat{\nu}} d\hat{r} = \int_{int}^{ext} t_{\hat{\mu}\hat{\nu}} \delta(\hat{r} - \hat{a}) d\hat{r}, \quad (56)$$

where  $\int_{int}^{ext}$  is an infinitesimal integral along the thin-shell. Using delta function property

$$\int g(x) \delta(x - x_o) dx = g(x_o), \quad (57)$$

we have

$$t_{\hat{\mu}\hat{\nu}} = \int_{int}^{ext} G_{\hat{\mu}\hat{\nu}} d\hat{r}. \quad (58)$$

*a. Surface Pressure* Now we will consider the surface energy density and surface tangential pressure terms. From (10) we can see that the  $G_{\hat{t}\hat{t}}$  component depends only on the first derivatives of the metric. Therefore, the surface energy density is

$$\Sigma = t_{\hat{t}\hat{t}} = \int_{int}^{ext} G_{\hat{t}\hat{t}} d\hat{r}. \quad (59)$$

When making the integration we will only obtain functions of the metric, and they are continuous because of the continuity condition for the metric. Since the integral is evaluated in the interior and exterior regions, the final integral vanishes. Hence, we have

$$\Sigma = 0. \quad (60)$$

On the other side, from equations (10) we see that the  $G_{\hat{\varphi}\hat{\varphi}}$  component has terms that depend on the first derivatives of the metric, and they will not contribute to the total integral. However, there is also a term with the form  $(1 - \frac{b}{r}) \Phi''$ . This term doesn't cancel out, and therefore, the surface tangential pressure can be written as

$$\mathcal{P} = t_{\hat{\varphi}\hat{\varphi}} = \int_{int}^{ext} G_{\hat{\varphi}\hat{\varphi}} d\hat{r} \quad (61)$$

$$\mathcal{P} = \left[ \sqrt{1 - \frac{b(a)}{a}} \Phi' \Big|_{int}^{ext} \right]. \quad (62)$$

Since we assume that a static internal observer does not feel any tidal force, we have  $\Phi'^{int} = 0$ . We also have

$$\Phi'^{ext} = \frac{a}{l^2} \left( -M + \frac{a^2}{l^2} \right)^{-1}. \quad (63)$$

Using (51) we obtain

$$\Phi'^{ext} = \frac{\frac{a}{l^2}}{\left( 1 - \frac{b(a)}{a} \right)}, \quad (64)$$



and then, the surface tangential pressure is

$$\mathcal{P} = \frac{\frac{a}{l^2}}{\sqrt{1 - \frac{b(a)}{a}}} \quad (65)$$

$$\mathcal{P} = \frac{\frac{a}{l^2}}{\sqrt{-M + \frac{a^2}{l^2}}}. \quad (66)$$

Note that in this case, the tangential pressure is always positive, under the condition

$$a^2 \geq Ml^2, \quad (67)$$

that corresponds to say that the wormhole's mouth is outside the event horizon correspondent to the exterior BTZ metric.

*b. Radial Pressure* The radial component of the field equations (19) let us write for the interior and exterior regions the expressions

$$\tau^{int}(r) = - \left( 1 - \frac{b^{int}}{r} \right) \frac{\Phi'^{int}}{r} - \Lambda^{int} \quad (68)$$

$$\tau^{ext}(r) = - \left( 1 - \frac{b^{ext}}{r} \right) \frac{\Phi'^{ext}}{r} - \Lambda^{ext}. \quad (69)$$

Since we assumed that interior static observers feel no tidal forces,  $\Phi'^{int}(a) = 0$ , we obtain

$$\tau^{int}(r) = -\Lambda^{int} \quad (70)$$

$$\tau^{ext}(r) = - \left( 1 - \frac{b^{ext}}{r} \right) \frac{\Phi'^{ext}}{r} - \Lambda^{ext} \quad (71)$$

Using equation (64) for  $\Phi'^{ext}$  and the tangential pressure given by (65), we have

$$\tau^{ext}(a) = -\frac{\mathcal{P}}{a} \sqrt{-M + \frac{a^2}{l^2}} - \Lambda^{ext} \quad (72)$$

So, this last equation gives a relation between the radial tension at the surface and the tangential pressure of the thin-shell.

## VII. SPECIFIC WORMHOLE SOLUTIONS

It is possible to define various functions that represent wormholes. In general, these solutions could have thin-shells or simply boundary surfaces. In any case, we will assume, from now on, that  $\Phi'^{int} = 0$  in order to permit the traversability of the wormhole.

### A. Junction with $\mathcal{P} = 0$ (Boundary Surface)

The exterior solution is vacuum with a negative cosmological constant, so we have  $\tau_{ext} = 0$  and  $\Lambda_{ext} = -\frac{1}{l^2} < 0$ . If we consider the boundary surface case, i. e.  $\mathcal{P} = 0$ , equation (72) gives

$$\Lambda^{ext} = 0. \quad (73)$$

This fact shows that there is no wormhole solution with  $\mathcal{P} = 0$  in universes with a negative cosmological constant (i.e. we can not have a BTZ solution outside).

### B. Junction with $\mathcal{P} \neq 0$ (Thin-Shell)

Again we have the conditions  $\tau_{ext} = 0$  and  $\Lambda_{ext} < 0$  but now we will consider a thin-shell, i. e.  $\mathcal{P} \neq 0$ . Now, equation (72) gives

$$\frac{\mathcal{P}}{a} \sqrt{-M + \frac{a^2}{l^2}} = -\Lambda^{ext} = \frac{1}{l^2} \quad (74)$$

and the form function given by (53), is now

$$b(a) = (1 + M)a - \frac{a^3}{l^2} \quad (75)$$

and then, the wormhole mass (54) is

$$M = \frac{b(a)}{a} + \frac{a^2}{l^2} - 1. \quad (76)$$

Is is clear that the mass associated with the wormhole is zero when  $b(a) = a - \frac{a^3}{l^2}$ , it is positive when  $b(a) > a - \frac{a^3}{l^2}$  and it is negative if  $b(a) < a - \frac{a^3}{l^2}$ .

For the subsequent steps, we will consider the limit case  $b(a) = a - \frac{a^3}{l^2}$ . Choosing the form function we will obtain different wormholes. Here, we consider only two possible functions.

1. First, consider the functions

$$\begin{aligned} b(r) &= (r_m r)^{\frac{1}{2}} \\ \Phi(r) &= \Phi_o \end{aligned} \quad (77)$$

where  $r_m$  is the throat radius. We have

$$\begin{aligned} b'(r) &= \frac{1}{2} \sqrt{\frac{r_m}{r}} \\ \Phi'(r) &= 0. \end{aligned} \quad (78)$$

The field equations (37) to (39) can be written as

$$\bar{\rho}(r) \equiv \rho(r) + \Lambda_{int} = -\frac{1}{4r^3} \sqrt{r_m r} \quad (79)$$

$$\bar{\tau}(r) \equiv \tau(r) + \Lambda_{int} = 0 \quad (80)$$

$$\bar{p}(r) \equiv p(r) - \Lambda_{int} = 0. \quad (81)$$

Note that in this case the matter density  $\rho$  can be negative, positive or zero, depending on the value of the internal cosmological constant  $\Lambda_{int}$ . The total matter density  $\bar{\rho}$  is always negative and correspond to the function shown in the Figure 1.

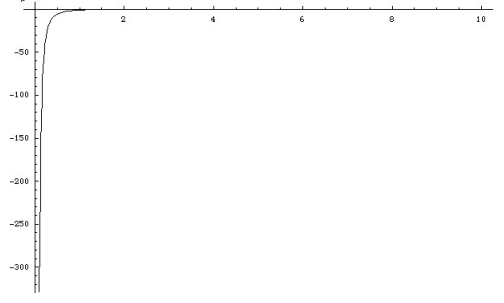


Figure 1. Mass density for the first wormhole solution. Note that this corresponds to exotic matter, and it is distributed in all the wormhole.

Equation (75) gives

$$b(a) = (r_m a)^{\frac{1}{2}} = a - \frac{a^3}{l^2} \quad (82)$$

$$\frac{a^2}{l^2} = 1 - \left(\frac{r_m}{a}\right)^{\frac{1}{2}}, \quad (83)$$

and in order to obtain a wormhole and not a black hole, we must impose  $a > r_+$ , that gives

$$a > \frac{r_m}{(M-1)^2}. \quad (84)$$

Moreover, the constant  $\Phi_o$  must satisfy  $e^{2\Phi(a)} = -M + \frac{a^2}{l^2}$  (equation 50). Therefore

$$e^{2\Phi_o} = -M + \frac{a^2}{l^2} \quad (85)$$

Finally, the metric is: in the interior, ( $r_m \leq r \leq a$ ),

$$ds^2 = -\left(-M + \frac{a^2}{l^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \sqrt{\frac{r_m}{r}}\right)} + r^2 d\varphi^2 \quad (86)$$

while in the exterior, ( $a \leq r \leq \infty$ ), the metric is

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2}\right)} + r^2 d\varphi^2. \quad (87)$$

2. Our second option of wormhole is

$$\begin{aligned} b(r) &= \frac{r_m^2}{r} \\ \Phi(r) &= \Phi_o \end{aligned} \quad (88)$$

with  $r_m$  the throat radius. Now, we have

$$\begin{aligned} b'(r) &= -\frac{r_m^2}{r^2} \\ \Phi'(r) &= 0. \end{aligned} \quad (89)$$

The field equations are now given by

$$\bar{\rho}(r) \equiv \rho(r) + \Lambda_{int} = \frac{1}{2r^3} \left[ -\frac{r_m^2}{r^2} r - \frac{r_m^2}{r} \right] = -\frac{r_m^2}{r^5} \quad (90)$$

$$\bar{\tau}(r) \equiv \tau(r) + \Lambda_{int} = 0 \quad (91)$$

$$\bar{p}(r) \equiv p(r) - \Lambda_{int} = 0. \quad (92)$$

Note that the value of the mass density  $\rho$  is negative, positive or zero depending on the value of the internal cosmological constant  $\Lambda_{int}$ , while the total mass density  $\bar{\rho}$  is always negative and behaves like is shown in Figure 2.

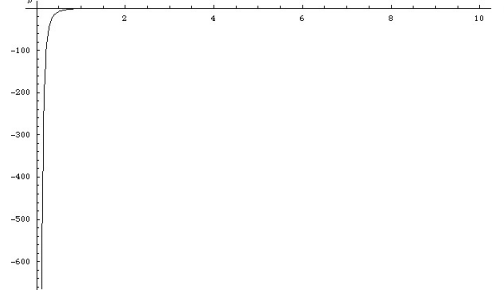


Figure 2. Mass density for the second wormhole solution. Again, this corresponds to exotic matter, and it is distributed in all the wormhole.

Using equation (??) we obtain

$$b(a) = \frac{r_m^2}{a} = a - \frac{a^3}{l^2} \quad (93)$$

$$r_m^2 = a^2 - \frac{a^4}{l^2}. \quad (94)$$

So, the wormhole mouth must be located at

$$a^2 = \frac{l^2}{2} \left[ 1 \pm \sqrt{1 - 4 \frac{r_m^2}{l^2}} \right]. \quad (95)$$

To obtain a wormhole solution and not a black hole we must impose the condition  $a > r_+$ , that gives

$$1 \pm \sqrt{1 - 4 \frac{r_m^2}{l^2}} > 2M. \quad (96)$$

The constant  $\Phi_o$  must satisfy  $e^{2\Phi(a)} = -M + \frac{a^2}{l^2}$  again (equation 50). Then

$$e^{2\Phi_o} = -M + \frac{a^2}{l^2}. \quad (97)$$

Finally, the metric of the wormhole is, in the interior  $r_m \leq r \leq a$ ,

$$ds^2 = - \left( -M + \frac{a^2}{l^2} \right) c^2 dt^2 + \frac{dr^2}{\left( 1 - \frac{r_m^2}{r^2} \right)} + r^2 d\varphi^2 \quad (98)$$

and in the exterior,  $a \leq r \leq \infty$ ,

$$ds^2 = - \left( -M + \frac{r^2}{l^2} \right) dt^2 + \frac{dr^2}{\left( -M + \frac{r^2}{l^2} \right)} + r^2 d\varphi^2. \quad (99)$$

### VIII. CONCLUSION

In this paper we have consider the usual method for construction of wormholes, by joining two spacetimes in the formalism of  $(2+1)$  dimensional gravity with a negative cosmological constant. In the internal region we impose a appropriate geometry to obtain a traversable wormhole, while, in the exterior region we use a BTZ black hole solution. In this way we obtain two specific representing traversable wormholes. It is also shown that both solutions need of some exotic matter to exist.

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